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Stiffness-Matrix Condition Number and Shape Sensitivity Errors

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Introduction

FOR static response, the condition number of the stiffness matrix is an upper bound to the amplification of errors in structural properties and loads. However, even though typical stiffness matrices have condition numbers larger than one million, we do not expect that errors or variations in the structure or loads would be amplified so much. The present Note seeks to explain why in most cases our expectation is fulfilled. It also presents an example of a case associated with shape sensitivity analysis where the worst-case scenario predicted by the condition number is much closer to the actual error amplification. A criterion is proposed that is closer to the actual error magnification than the condition number.

Consider the discretized equations of equilibrium of static response such as those generated by a finite-element model

$$Ku = f \tag{1}$$

where K is the $n \times n$ symmetric, positive, definite, stiffness matrix, u the displacement vector, and f the load vector. The condition number of K, $c(K)$ is defined as

$$c(K) = \|K\| \; \|K^{-1}\| \tag{2}$$

when the 2-norm is used

$$c(K) = \lambda_n / \lambda_1 \tag{3}$$

where λ_i denotes the ith eigenvalue of K. It is well known (e.g., see Ref. 1) that $c(K)$ is an upper bound on the sensitivity of u to perturbations in K and f. That is, if we perturb f by Δf then

$$\frac{\|\Delta u\|}{\|u\|} \leq c(K) \frac{\|\Delta f\|}{\|f\|} \tag{4}$$

and if we perturb K by ΔK then

$$\frac{\|\Delta u\|}{\|u + \Delta u\|} \leq c(K) \frac{\|\Delta K\|}{\|K\|} \tag{5}$$

The condition number for most stiffness matrices generated by finite-element models runs into the millions. This would appear to indicate that the computed displacement field can be extremely sensitive to small errors in the stiffness matrix and force vectors. In spite of this theoretical sensitivity, we continue to approximate the stiffness matrix (e.g., by reduced integration) and the force vector (e.g., lumping loads) without fear of the huge amplification of errors predicted by the condition number. It is known, in fact, that the condition number may be an overly conservative estimate of error sensitivity.[2]

The condition number is particularly overconservative for predicting sensitivity to changes in the load vector. For a given K and f, it is always possible to find a ΔK to make Eq. (5) an equality. Also, $c(K)$ can be a good predictor of roundoff error amplification so that if the condition number is 10^7 and we work with 7-digit numbers, the errors in u can be very large. In the following it is assumed that the number of digits available for computation is much larger than the condition number (a typical case in finite-element computation is $c(K) = 10^7$ with 15-digit computations). However, it is not usually possible to find a Δf to make Eq. (4) an equality.[2] The present Note derives a sharper estimate for sensitivity to load errors. It also presents a case where the extreme sensitivity predicted by the condition number is more closely realized.

Error Analysis

Let the eigenvectors of K be denoted as u_i, $i = 1,\ldots,n$ normalized to $\|u_i\| = 1$ with λ_i being the corresponding eigenvalues. We expand the load vector in terms of the eigenvectors as

$$f = \sum_{i=1}^{n} \alpha_i u_i \tag{6}$$

and similarly the perturbation or error in the load as

$$\Delta f = \sum_{i=1}^{n} \Delta\alpha_i u_i \tag{7}$$

It is then easy to check that u can be obtained as

$$u = \sum_{i=1}^{n} (\alpha_i / \lambda_i) u_i \tag{8}$$

with a similar expansion for Δu. The error amplification factor e is defined as

$$e = \frac{\|\Delta u\|}{\|\Delta u\|} \bigg/ \frac{\|\Delta f\|}{\|f\|} \tag{9}$$

Using the orthonormality of the eigenvectors we get

$$e^2 = \frac{(\Sigma_{i=1}^{n} \Delta\alpha_i^2 / \lambda_i^2)(\Sigma_{i=1}^{n} \alpha_i^2)}{(\Sigma_{i=1}^{n} \alpha_i^2 / \lambda_i^2)(\Sigma_{i=1}^{n} \Delta\alpha_i^2)} \tag{10}$$

It is easy to check that the worst case is when the perturbation is in the shape of the first eigenvector, $\Delta f = \Delta\alpha_1 u_1$ so that an upper bound on e, called here the error magnification index

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and denoted e_m, is given as

$$e_m = \frac{1}{\lambda_1}\frac{\|f\|}{\|u\|} \tag{11}$$

Equation (11) predicts that the error amplification is large when $\|f\|$ is large and $\|u\|$ is small. This will happen when f is in the shape of a combination of higher eigenvectors [see Eqs. (6) and (8)]. For example f can be highly oscillatory in its spatial distribution or in a high, aspect-ratio, beam-type structure, f could correspond to shear loading. For $f = \alpha_n u_n$ and $\Delta f = \Delta\alpha_i u_1$, we get

$$e = e_m = \lambda_n/\lambda_1 = c(K) \tag{12}$$

so that when the load is in the shape of the last eigenvector, and the load error is in the shape of the first eigenvector, the error amplification is indeed equal to the condition number.

The error magnification index e_m given by Eq. (11) is a much sharper estimate of error amplification than the condition number $c(K)$. In fact, for a force in the shape of the first eigenvector, $f = \alpha_1 u_1$, Eq. (11) gives $e_m = 1$, so that there is no error amplification no matter how high the condition number. Because in most practical situations, f is a linear combination of the first few eigenvectors, e_m is much smaller than $c(K)$, and we do not get large force-error magnification even when the condition number is high.

Application to Shape Sensitivity

We can obtain the sensitivity derivative u' of the displacement with respect to a structural parameter v by differentiating Eq. (1) as

$$Ku' = -K'u \equiv f_p \tag{13}$$

where a prime denotes a derivative, and it is assumed that f does not depend on structural parameter. The right side of Eq. (13) f_p is called the pseudoload, and it is often approximated as

$$f_p = -K'u \cong -\frac{K(v+\Delta v) - K(v)}{\Delta v}u \tag{14}$$

that is by a forward finite-difference approximation. This approach to calculating u' is called the semianalytic method and is implemented in NASTRAN and other finite-element programs. For beam and platelike structures a semianalytical method can result in large errors due to the approximation of Eq. (14) when v is a shape parameter.[3,4] This behavior is now analyzed with the aid of the error magnification factor index of Eq. (11).

Consider the cantilever beam shown in Fig. 1. The beam is divided into m finite elements (which yield the exact solution for any m). The components of u are the normal displacement w_i and slope θ_i at the nodes given by

$$w_i = \frac{Mx_i^2}{2EI} = \frac{Mi^2L^2}{2EIm^2}, \quad \theta_i = \frac{Mx_i}{EI} = \frac{MiL}{mEI}, \quad i = 0,1,\ldots,m \tag{15}$$

The derivatives of w_i and θ_i with respect to L are

$$w_i' = \frac{Mi^2L}{EIm^2} = \frac{Mx_i^2}{EIL}, \qquad \theta_i' = \frac{Mi}{mEI} = \frac{Mx_i}{EIL} \tag{16}$$

As was noted in Ref. 4, w' and θ' are not matched in the sense that θ' is not the slope of w'.

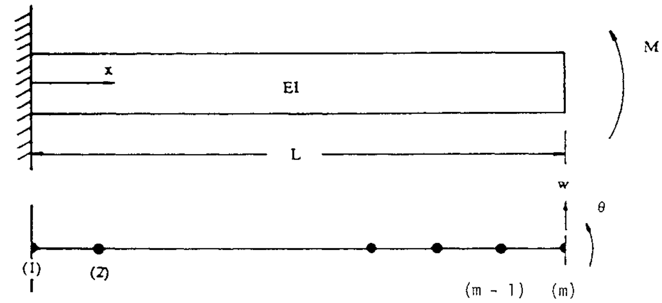

Fig. 1 Geometry, loading, and discretization for cantilever beam.

Table 1 Dependence of condition number, error magnification index, and actual errors in semianalytical, tip-rotation derivatives on the number of elements

		Derivative w.r.t.L		Derivative w.r.t.h	
Number of elements	Condition number	e_d[a]	Percent error in tip rotation	e_d[a]	Percent error in tip rotation
1	19.3	5.67	4	1.1	1.0
2	420	49.1	16	1.3	1.0
3	2200	135	35	1.5	1.0
4	6900	273	62	1.7	1.0
5	16,660	472	97	1.9	1.0
6	34,000	737	140	2.0	1.0
7	62,400	1076	190	2.2	1.0
8	105,000	1500	249	2.3	1.0
9	167,000	2000	314	2.4	1.0
10	253,000	2590	388	2.5	1.0

[a] $e_d = \frac{1}{\lambda_1}\frac{\|f_p\|}{\|u'\|}$

In fact, as m goes to infinity, we get from Eq. (16) that

$$\frac{\partial w'}{\partial x} = 2\theta' \tag{17}$$

That is, the derivatives of the displacements and rotations are mismatched in that the derivative of the slope is only one half of the slope of the derivative. This mismatch between w' and θ' results in a u' representing a displacement shape where each element is being bent into an s-shape, no matter how many elements we have. When m is large, this is a short-wave displacement shape that would be represented by the last few eigenvectors of K. The error analysis of the previous section would then predict the potential for large error magnification—especially for long-wave errors. The error magnification index for the derivative e_d is defined based on Eqs. (11) and (13) as

$$e_d = \frac{1}{\lambda_1}\frac{\|f_p\|}{\|u'\|} \tag{18}$$

The large errors associated with the semianalytical method for shape derivatives are in contrast to the small errors for size or stiffness derivatives. To show that the error magnification index discriminates between the two, we compare the derivative with respect to the length of the beam with the derivative with respect to the height h of the cross section of the beam (assumed to have a rectangular cross section). The pseudoload f_p is calculated from Eq. (14) for a change in length or height corresponding to 1% of the nominal value. The error in f_p is then of the order of 1%.

The effect of number of elements on the error magnification index and the actual error is shown in Table 1. For derivatives with respect to length, e_d increases rapidly with the number of elements lagging behind the condition number by about one to two orders of magnitude. The actual error also increases fast though not as fast as e_d. The potential error magnification of e_d is not realized because the error in f_p due to the finite-difference approximation of Eq. (14) is not in the shape of the lowest eigenvector [which is the worst-case scenario assumed in Eq. (11)].

The error magnification index for the height derivative increases very slowly with the number of elements and predicts well the sensitivity of that derivative to errors in the pseudoload. A 1% perturbation in a design variable for the purpose of derivative calculation is too large for most practical examples. However, for this example, the errors in the semianalytical method are almost exactly proportional to the perturbation so that the errors in Table 1 (columns 4 and 6) simply scale as the perturbation size is scaled.

Concluding Remarks

An error magnification index was proposed for assessing the sensitivity of the displacement field to errors in the load vector.

The index is less conservative than the condition number of the stiffness matrix and reflects the fact that for some cases, no error magnification occurs even when the condition number is very high. The proposed index was applied to calculation of derivatives of beam response to changes in the beam structural parameters by the semianalytical method. The calculation of derivative with respect to length is very sensitive to errors; whereas the calculation of derivative with respect to cross-sectional height is not. The proposed index indiscriminated well between these two cases.

Acknowledgments

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Accuracy of Condensed Eigenvalue Solution

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I. Introduction

A LARGE number of degrees of freedom are used to model complex structures. Efficient solution of the resulting eigenvalue problem is of the utmost importance for dynamic analysis. One way of dealing with complex structures is to partition the structure into a number of substructures. Lower natural frequencies and modal vectors of the complete structure can be calculated using the Guyan reduction method. In this case, the boundary nodes of substructures determine the master degrees of freedom for the condensation. The accuracy of the natural frequencies calculated in this manner are determined by the relative magnitude of the natural frequencies of the substructures corresponding to fixed boundary degrees of freedom. The substructure natural frequencies should be very large compared to the global natural frequencies to be calculated; otherwise, the solution of a frequency-dependent eigenvalue problem is necessary in order to obtain natural frequencies accurately. In this Note, a computationally efficient method is presented for the solution of this nonlinear eigenvalue problem.

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Condensation of Stiffness and Mass Matrices

Free vibration analysis of structures results in a matrix equation of the form

$$[K]\{U\}-\lambda[M]\{U\}=\{0\} \tag{1}$$

where $[K]$ and $[M]$ are the stiffness and mass matrices, respectively, $\{U\}$ is the modal vector, and λ is the square of the natural frequency. If this system of equations is partitioned, separating the boundary and internal degrees of freedom, the following equation system is obtained:

$$\begin{bmatrix} K_{BB} & K_{BI} \\ K_{IB} & K_{II} \end{bmatrix}\begin{Bmatrix} U_B \\ U_I \end{Bmatrix}-\lambda\begin{bmatrix} M_{BB} & M_{BI} \\ M_{IB} & M_{II} \end{bmatrix}\begin{Bmatrix} U_B \\ U_I \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{2}$$

From the second row of these matrix equations,

$$U_I=-(K_{II}-\lambda M_{II})^{-1}(K_{IB}-\lambda M_{IB})U_B \tag{3}$$

Here, we consider the complete solution of the eigenvalue problem in which only the internal degrees of freedom are present:

$$(K_{II}-\lambda M_{II})W=0 \tag{4}$$

where Λ is a diagonal matrix having substructure eigenvalues on its diagonal, and W is the substructure modal matrix. Using the identities

$$W^TM_{II}W=I \tag{5a}$$

$$W^TK_{II}W=\Lambda \tag{5b}$$

Eq. (3) can be rewritten as

$$U_I=-W(\Lambda-\lambda I)^{-1}W^T(K_{IB}-\lambda M_{IB})U_B \tag{6}$$

where I is the identity matrix.

The first row of Eq. (2) gives

$$(K_{BB}-\lambda M_{BB})U_B+(K_{BI}-\lambda M_{BI})U_I=0 \tag{7}$$

Equation (6) is substituted in Eq. (7) to give

$$(K_{BB}-\lambda M_{BB})U_B - (K_{BI}-\lambda M_{BI})W(\Lambda-\lambda I)^{-1}W^T(K_{IB}-\lambda M_{IB})U_B=0 \tag{8}$$

or

$$D(\lambda)U_B=0$$

Expansion of the second term in Eq. (8) results in the following expression for the dynamic stiffness matrix:

$$D(\lambda)=K_0-\lambda M_0-\lambda G(\Lambda-\lambda I)^{-1}G^T \tag{9}$$

where

$$K_0=K_{BB}-K_{BI}K_{II}^{-1}K_{IB}$$

$$M_0=M_{BB}+K_{BI}K_{II}^{-1}M_{II}K_{II}^{-1}K_{IB}-K_{BI}K_{II}^{-1}M_{IB}-M_{BI}K_{II}^{-1}K_{IB}$$

$$G=M_{BI}W-K_{BI}W\Lambda^{-1}$$

In Guyan reduction, the last term in Eq. (9) is neglected. Hence, a linear eigenvalue problem is obtained. The effect of this term on the accuracy of the eigenvalues is determined by the magnitude of the diagonal entries

$$\lambda/(\lambda_k-\lambda)$$

where λ_k are the substructure eigenvalues obtained from Eq. (4). It is shown in Ref. 1 that the error in the ith eigenvalue is